Comment on "Stability of Tsallis entropy and instabilities of Rényi and normalized Tsallis entropies: A basis for q-exponential distributions"

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It is shown that the Rényi entropy is as stable as the Tsallis entropy at least for the Abe-Lesche counter examples.

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Abe [1] presented two counterexamples of instability of the Rényi entropy and showed that the Tsallis entropy is stable for these counterexamples. From the time of its publication, this work is often referred (see, e.g., [2,3]) as a mortal verdict for the Rényi entropy. On the other hand, the Rényi entropy is widely used now. Because of this, the main points of Ref. [1] are to be revised carefully.

Abe calculated responses $|\Delta S^{(R)}|$ and $|\Delta S^{(Ts)}|$ to small variations of initial model distributions over W states of a system and then passed to the limit $W\to\infty$, treating an amplitude δ of the variation as a finite constant. As a result, he found a loss of continuity of a response of the Rényi entropy to the small perturbations. In my point of view, such a conclusion can be dismissed on two counts. First, Abe considered normalized values of $|\Delta S^{(R)}|$ and $|\Delta S^{(Ts)}|$ with different W-dependent normalization factors, $S^{(R)}_{max}$ and $S^{(Ts)}_{max}$, correspondingly. Such normalization influenced their limiting properties and, consequently, conclusions about their stabilities. Second, continuity is to be checked with the use of the opposite iterated limiting process: firstly, $\delta\to 0$ and then $W\to\infty$. Such an order corresponds to a traditional approach in statistical physics where all properties are calculated firstly for finite systems and the thermodynamic limit is performed after all calculations (see, e.g., [4]). Below are modifications of Abe's results for such order of the limiting procedures.

For brevity's sake, the first of Abe's counterexamples alone will be discussed here. It is especially important, because it refers to the range $0<q<1$ the most, if not all, of applications [5] of the Rényi entropy. The second counterexample may be discussed in the same manner.

The examined small ($\delta\ll 1$) deformation of distribution $\{p\}$ over W states ($W\gg 1$) for $0<q<1$ is

$$p_i=\delta_{i1},\quad p_i'=\left(1-\frac{\delta}{2}\frac{W}{W-1}\right)p_i+\frac{\delta}{2}\frac{1}{W-1}. \tag{1}$$

Using the well-known definitions of the Tsallis and Rényi entropies (for $k_B=1$), we get

$$|\Delta S^{(Ts)}|=\frac{1}{1-q}\left[\left(1-\frac{\delta}{2}\right)^q+\left(\frac{\delta}{2}\right)^q(W-1)^{1-q}-1\right], \tag{2}$$

$$|\Delta S^{(R)}|=\frac{1}{1-q}\ln\left[\left(1-\frac{\delta}{2}\right)^q+\left(\frac{\delta}{2}\right)^q(W-1)^{1-q}\right]<|\Delta S^{(Ts)}|, \tag{3}$$

where the last inequality is resulted from the fact that the logarithm as a concave function is always less than its linearized approximation. Thus, stability of the Rényi entropy for the counterexample (1) is at least not lower than the stability of the Tsallis entropy.

I may suppose that Abe paid no attention to this evident inequality because he was developing Lesche-stability conditions [6], which are not for $|\Delta S^{(R)}|$ and $|\Delta S^{(Ts)}|$, but for

$$\left|\Delta\left(\frac{S^{(R)}}{S^{(R)}_{max}}\right)\right| \quad\text{and}\quad \left|\Delta\left(\frac{S^{(Ts)}}{S^{(Ts)}_{max}}\right)\right|,$$

where $S^{(R)}_{max}=\ln W$ and $S^{(Ts)}_{max}=(W^{1-q}-1)/(1-q)$. Their ratio at large W is

$$\frac{S^{(Ts)}_{max}}{S^{(R)}_{max}}=\frac{1}{1-q}\left.\frac{W^{1-q}-1}{\ln W}\right|_{W\to\infty}\to W^{1-q}. \tag{4}$$

It is just this additional multiplier that ensures convergence of the gain of the normalized Tsallis entropy $S^{(Ts)}/S^{(Ts)}_{max}$ in contrast to $S^{(R)}/S^{(R)}_{max}$.

It would be more reasonable to normalize both entropies by the common denominator, say, $\sup\{S^{(R)}_{max},S^{(Ts)}_{max}\}$ or number of states W. In the latter case, $|\Delta S|/W$ is the entropy gain per a state. It is evident from the above equations that the gain per a state for each of the discussed entropies tends to zero when $W\to\infty$.

But the most essential point is that we should only deal with entropies as such, because their normalized versions $S^{(R)}/S^{(R)}_{max}$ and $S^{(Ts)}/S^{(Ts)}_{max}$ are irrelevant both to thermostatistics and information theory.

As for stability of the Rényi and Tsallis entropies *per se*, their gains for $\delta\ll 1$ and $W\gg 1$ become

$$|\Delta S^{(Ts)}(p)|\simeq\frac{1}{1-q}\left(\frac{\delta}{2}\right)^q(W-1)^{1-q},\quad 0<q<1, \tag{5}$$

$$|\Delta S^{(R)}(p)|\lesssim\frac{1}{1-q}\left(\frac{\delta}{2}\right)^q(W-1)^{1-q},\quad 0<q<1, \tag{6}$$

irrespective of the interrelation between δ and W.

Because both entropies are unbounded functionals, there

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is no double limit ($\delta \to 0$, $W \to \infty$) of their gain as a function of δ and W, but there is a repeated limit ($\delta \to 0$ and then $W \to \infty$) and it is equal to zero. Indeed, both ΔS become infinitesimal for any finite W when

$$\delta/2 \ll (W-1)^{-(1-q)/q}. \tag{7}$$

In terms of ε-δ, the continuity condition is formulated in the next form: For every given $\varepsilon > 0$ we are to find such δ that both ΔS become less ε if $\Sigma_i |p_i - p_i'| \leqslant \delta$. Here it means that

$$\delta < 2[(1-q)W^{q-1}\varepsilon]^{1/q}, \quad 0 < q < 1. \tag{8}$$

Dependence of this condition on W testifies that both $S^{(R)}$ and $S^{(Ts)}$ are not uniformly convergent when $W \to \infty$.

At any case there are no advantages of the Tsallis entropy over Rényi entropy for its stability in a sense of continuity relative to small perturbation of the probability distribution.

[1] S. Abe, Phys. Rev. E **66**, 046134 (2002).
[2] C. Anteneodo and C. Tsallis, J. Math. Phys. **44**, 5194 (2003).
[3] S. Kaniadakis and A. M. Scarfone, Physica A **340**, 102 (2004).
[4] A. Isihara, *Statistical Physics* (Academic, New York, 1971).
[5] A. G. Bashkirov, Phys. Rev. Lett. **93**, 130601 (2004).
[6] B. J. Leshe, J. Stat. Phys. **27**, 419 (1982).